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On Noncharacteristic Boundary Conditions for Discrete Hyperbolic Initial-Boundary-Value Problems

W. M. COUGHRAN, JR.

Computing Mathematics Research Department, AT & T Bell Laboratories, Murray Hill, New Jersey 07974

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The stability properties of some discrete hyperbolic initial-boundary-value problems, involving two dependent variables and noncharacteristic boundary conditions, are examined. Approximations based on the Lax–Wendroff, leap-frog, and Crank–Nicolson methods are considered. The most accurate approximations are found to be the least stable. The results for the model problems can be used in the stability analysis of larger problems where characteristics occur singly or in pairs; in particular, discretizations of the shallow-water and gas-dynamics equations, which involve boundary conditions written in natural variables, are considered. Some computations demonstrating the stability results are discussed. An appendix discussing numerical stability-verification techniques is included. © 1985 Academic Press, Inc.

1. INTRODUCTION

The paper of Gustafsson, Kreiss, and Sundström [1] provides a theory for verifying the stability of discrete hyperbolic initial-boundary-value problems (IBVPs). Their approach is often called normal-mode analysis. (Some knowledge of the continuous theory and normal-mode analysis is assumed [2–5]; the notion of stability used here is given by Definition 3.3 of [1].) For scalar IBVPs, normal-mode analysis has been used to study many interior/boundary schemes.

Establishing the stability of difference approximations to vector IBVPs via normal-mode analysis is arduous, in general. Several authors have considered the stability of discretizations of first-order hyperbolic systems corresponding to the classical wave equation and not written in the characteristic variables [1, 6–12]. It is known that interior/boundary approximation combinations that are stable for scalar problems are not always stable for vector problems [9]. However, if the incoming and outgoing characteristics are handled separately at the boundary, verifying the stability of discrete vector IBVPs reduces to the analysis of scalar IBVPs [13, 14].

It is more natural to work in the variables that the differential equations are usually written in, although such *natural* variables may have both incoming and outgoing characteristic components. Normal-mode analysis is cumbersome to apply

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in such situations but the natural variables are often preferred by practitioners. (A boundary correction procedure is possible for explicit schemes [14], and is generally required, but is not always used.) Finally, the linear equations arising from implicit discretizations may be more complicated in the characteristic variables.

The emphasis here is on the stability of discrete IBVPs that are not in characteristic form (unlike [13, 14]). By studying simple model problems, some classical fluid-dynamics systems may be dealt with since characteristics often occur singly and in pairs.

Normal-mode analysis is used, in Section 2, to investigate the stability of some discrete IBVPs, involving two dependent variables; characteristic variables are not used in formulating boundary conditions. The most accurate approximations are found to be the least stable. The Lax–Wendroff method is seen to be stable with several boundary schemes while the nondissipative leap-frog and Crank–Nicolson methods are unstable with most boundary schemes. However, dissipativity is seen to be insufficient to guarantee stability [9].

Sections 3 and 4 consider the stability of discretizations of the shallow-water and gas-dynamics equations. Section 5 discusses some computations that illustrate the stability analyses. Section 6 offers some conclusions and the Appendix discusses numerical stability verification.

2. An Off-Diagonally Dominant Model Problem

Consider an off-diagonally dominant model problem, involving two dependent variables,

$$w_t = Aw_x = \begin{bmatrix} b & 1 \\ 1 & b \end{bmatrix} w_x \qquad \text{for} \quad (x, t) \in \mathbb{R}_+^2$$

$$w(x, 0) = w_I(x) \qquad \text{for} \quad x \in \mathbb{R}_+$$

$$u(0, t) = d(t) \qquad \text{for} \quad t \in \mathbb{R}_+$$

where $w = (u, v)^T$ and $|b| < 1$. (This system is equivalent to $u_{tt} = u_{xx}$ when $b = 0$.)

The eigenvalues of A are $\mu_\pm = b \pm 1$. Hence, the problem is strictly hyperbolic and the $x = 0$ boundary is not characteristic so the normal-mode stability theory applies to discretizations of the problem. The matrix of right eigenvectors is

$$T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

and $T^{-1}AT = \Lambda = \text{diag}(\mu_-, \mu_+)$.

Let ω_- and ω_+ denote the incoming and outgoing, at $x = 0$, characteristic variables, respectively, and $\omega = (\omega_-, \omega_+)^T$. Then $w = T\omega = (\omega_+ + \omega_-, \omega_+ - \omega_-)^T$.

Note that specifying u at the boundary incorporates some reflection of the outgoing characteristic variable back into the problem domain.

The next three subsections present stability results for the Lax–Wendroff, leap-frog, and Crank–Nicolson methods with a variety of boundary schemes (more details are available in [15]). The last subsection discusses previous work, implications of the stability results presented here, and extensions to other model problems involving two variables.

2.1. *Lax–Wendroff*

The discrete analog based on Lax–Wendroff (LW) in the interior and extrapolation (EX) as the auxiliary boundary condition is

$$\tilde{w}_\nu(t+k) = [I + kAD_0 + \tfrac{1}{2}k^2A^2D_+D_-]\,\tilde{w}_\nu(t) \qquad \text{for} \quad \nu > 0 \tag{2.1}$$

$$\tilde{w}_\nu(0) = w_I(\nu h) \qquad \text{for} \quad \nu \geqslant 0 \tag{2.2}$$

$$\tilde{u}_0(t+k) = d(t+k) \tag{2.3}$$

$$D_+^j\,\tilde{v}_0(t+k) = 0 \tag{2.4}$$

where h and k are the spatial and temporal mesh spacings, respectively, $D_+ = (E-I)/h$, $D_- = (I-E^{-1})/h$, $D_0 = (D_+ + D_-)/2$, and $E\tilde{u}_\nu = \tilde{u}_{\nu+1}$. Let $\lambda = k/h$; in what follows, assume λ is sufficiently small to guarantee stability of the interior discretization for the associated Cauchy problem (Cauchy stability).

Without loss of generality, assume homogeneous initial and boundary conditions. Then the discrete problem in characteristic variables is

$$\tilde{\omega}_\nu(t+k) = [I + kAD_0 + \tfrac{1}{2}k^2A^2D_+D_-]\,\tilde{\omega}_\nu(t) \qquad \text{for} \quad \nu > 0$$

$$\tilde{\omega}_\nu(0) = 0 \qquad \text{for} \quad \nu \geqslant 0$$

$$\tilde{u}_0(t+k) = (\tilde{\omega}_+ + \tilde{\omega}_-)_0(t+k) = 0$$

$$D_+^j\,\tilde{v}_0(t+k) = D_+^j(\tilde{\omega}_+ - \tilde{\omega}_-)_0(t+k) = 0.$$

Let $\kappa_-(z)$ and $\kappa_+(z)$ denote the roots of the characteristic equation,

$$\det[\kappa(z-1)\,I - \tfrac{1}{2}\lambda(\kappa^2-1)\,A - \tfrac{1}{2}\lambda^2(\kappa-1)^2\,A^2] = 0,$$

that yield l_2 solutions for $|z| > 1$, respectively. The appropriate normal-mode trial solution is $\tilde{\omega}_\nu(t) = z^{t/k}(\rho_-\kappa_-^\nu, \rho_+\kappa_+^\nu)^T$ for some scalars $\rho_\pm$.

If this trial solution is substituted into the obvious boundary condition (2.3) with homogeneous data, the result is $\rho_- + \rho_+ = 0$. Similarly, $\rho_+(\kappa_+ - 1)^j - \rho_-(\kappa_- - 1)^j = 0$ comes from (2.4). Hence, the determinant condition for the existence of an eigenvalue or generalized eigenvalue is

$$\det\begin{bmatrix} 1 & 1 \\ -(\kappa_- - 1)^j & (\kappa_+ - 1)^j \end{bmatrix} = (\kappa_- - 1)^j + (\kappa_+ - 1)^j = 0. \tag{2.5}$$

(Recall that a generalized eigenvalue is when $|z| = 1$ and $|\kappa_{\pm}| = 1$.) The discrete initial-boundary-value problem (2.1)–(2.4) is stable if and only if (2.5) cannot be satisfied for $|z| \geqslant 1$ [1].

It is easy to show that

PROPOSITION. *Lax–Wendroff with zeroth-order* ($j = 1$) *extrapolation is stable for the problem* (2.1)–(2.4) *if* $\lambda\rho(A) \leqslant 1$, *where* $\rho(A)$ *denotes the spectral radius of* A.

Proof. For $j = 1$ the determinant condition is simply $\kappa_- + \kappa_+ = 2$. If the scalar equation $u_t = cu_x$ is approximated by a consistent centered second-order dissipative Cauchy-stable scheme (involving at most three points per time level) then for $|z| \geqslant 1$ there is a $\delta > 0$ such that $|\kappa| \leqslant 1$ if $c > 0$ and $|\kappa| \leqslant 1 - \delta$ if $c < 0$, where the κ is the root that yields l_2 solutions for $|z| > 1$ [16]. Hence, $|\kappa_-| \leqslant 1 - \delta$ and $|\kappa_+| \leqslant 1$ so the determinant condition cannot be satisfied. ∎

Unfortunately, EX with $j = 1$ is insufficiently accurate to insure second-order convergence to the solution in the interior [17].

Determinant stability conditions for vector problems are usually difficult to verify since detailed information about how κ_- and κ_+ are affected by changes in z is needed. The stability results presented later are based on computational verification of the determinant condition, as described in the Appendix. (Other authors have used numerical verification to study stability questions [18, 9, 10, 19, 20].) The use of numerical techniques to investigate stability introduces some uncertainties because of termination criteria, machine arithmetic, and other complications, but such procedures can deal with complex problems.

When computationally studying the determinant condition, it is necessary to make further restrictions on the off-diagonally dominant model problem. In particular, the two free parameters, b and λ, are taken from the discrete set

$$b := -0.5,\ -0.4, \ldots,\ -0.1,\ -0.05,\ -0.04, \ldots,\ -0.01, 0,$$
$$0.01, 0.02, \ldots, 0.05, 0.1, 0.2, \ldots, 0.5;$$
$$\lambda\rho(A) := 0.1, 0.2, 0.3, \ldots, 0.9.$$

There are 189 cases to check for each boundary scheme.

For a given value of b (and the set of λ's), any of three unstable eigenvalues are possible. Eigenvalues with $|z| = 1$ represent the weakest instability. Generalized eigenvalues are stronger instabilities than eigenvalues on the unit disk. Eigenvalues with $|z| > 1$ represent the most dramatic instabilities. Hence, in summarizing the results of the numerical normal-mode analysis for a particular b, the worst instability found for the various λ's is noted. In other words, if only eigenvalues with $|z| = 1$ are found, b is said to give rise to eigenvalues on the unit disk. If generalized eigenvalues, but no eigenvalues with $|z| > 1$, are found then b is said to yield generalized eigenvalues. Finally, if any eigenvalues with $|z| > 1$ are found, b is said to result in eigenvalues outside of the unit disk.

The results of numerical normal-mode analysis of (2.1)–(2.4) are summarized in Table I.

Schemes that approximate the second differential equation, $v_t = u_x + bv_x$, are natural replacements for the extrapolation condition on $\tilde{v}$ (2.4). For LW in the interior, it is reasonable to investigate the analogs of the Euler, box, and folded Lax–Wendroff schemes. These schemes are stable for scalar problems [1, 21], involve only one previous time level, and are sufficiently accurate to insure second-order convergence [17].

The analog of the Euler (EL) scheme is

$$\tilde{v}_0(t+k) = \tilde{v}_0(t) + kD_+[\tilde{u}_0(t) + b\tilde{v}_0(t)]. \tag{2.6}$$

The associated determinant condition is

$$\lambda\mu_-(\kappa_- - 1) + \lambda\mu_+(\kappa_+ - 1) - 2(z-1) = 0.$$

The results of numerical verification for (2.1)–(2.3) and (2.6) are summarized in Table I.

The box (BX) boundary scheme is

$$\begin{aligned} &\tilde{v}_0(t+k) + \tilde{v}_1(t+k) \\ &\quad = \tilde{v}_0(t) + \tilde{v}_1(t) + kD_+\{\tilde{u}_0(t+k) + \tilde{u}_0(t) + b[\tilde{v}_0(t+k) + \tilde{v}_0(t)]\}. \end{aligned} \tag{2.7}$$

The results of normal-mode analysis for (2.1)–(2.3) and (2.7) are summarized in Table I.

Finally, consider using the folded Lax–Wendroff (FLW) scheme at the boundary [21]. The appropriate transformation of variables is

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{bmatrix} 1 & -\cos\theta \\ 0 & \sin\theta \end{bmatrix} \begin{pmatrix} \xi \\ \tau \end{pmatrix}$$

TABLE I

Off-Diagonal Stability Results for Lax–Wendroff[a]

Boundary scheme	Stable	EV with $\lvert z\rvert = 1$	GEV	EV with $\lvert z\rvert > 1$
EX ($j=1, 2$), EL, BX	$-0.5 \leqslant b \leqslant 0.5$			
EX ($j=3$)	$0 < b \leqslant 0.5$	$b=0$		$-0.5 \leqslant b < 0$
EX ($j=4$)				$-0.5 \leqslant b \leqslant 0.5$
FLW	$0 < b \leqslant 0.5$ $-0.5 \leqslant b \leqslant -0.3$	$b=0$		$-0.2 \leqslant b < 0$

[a] Recall that the stability summary lists the computed range of b values for which an eigenvalue with $|z| > 1$, a generalized eigenvalue (GEV) with $|z| = 1$, an eigenvalue (EV) with $|z| = 1$, or no instability is found; a range specifications of the form $l \leqslant b \leqslant u$ only means that the b's taken from the discrete set satisfy the inequality.

where $0 < \theta = \tan^{-1} \lambda < \pi/2$. After some manipulation, the FLW boundary scheme is found to be

$$\tilde{v}_0(t+k) = \{I + (\lambda b - 1)(hD_0) + \tfrac{1}{2}[(\lambda b - 1)^2 + \lambda^2](h^2 D_+ D_-)\}\, \tilde{v}_1(t) + [\lambda(hD_0) + \lambda(\lambda b - 1)(h^2 D_+ D_-)]\, \tilde{u}_1(t). \tag{2.8}$$

The stability results for (2.1)–(2.3) and (2.8) appear in Table I.

2.2. *Leap-Frog*

The LW scheme (2.2) can be replaced by the second-order leap-frog (LF) method

$$\tilde{w}_\nu(t+k) = \tilde{w}_\nu(t-k) + 2kAD_0 \tilde{w}_\nu(t). \tag{2.9}$$

Let $\kappa_-(z)$ and $\kappa_+(z)$ denote the roots of the characteristic equation,

$$\det[\kappa(z^2 - 1)\, I - \lambda z(\kappa^2 - 1)\, A] = 0,$$

that yield l_2 solutions for $|z| > 1$, respectively.

With LF, it is reasonable to consider the EL (2.6) and BX (2.7) boundary conditions. The results of numerical verification for these schemes are shown in Table II.

Space-time extrapolation (ST)

$$^{-}D_+^j\, \tilde{v}_0(t+k) = (I - E_t^{-1} E_x)^j\, \tilde{v}_0(t+k) = 0 \tag{2.10}$$

can also be used. Numerical verification indicates that LF and ST with $j = 1$ form a stable approximation, which is insufficiently accurate to insure second-order convergence [17]. The results for LF and ST are summarized in Table II.

TABLE II

Off-Diagonal Stability Results for Leap-Frog

Boundary scheme	Stable	EV with $\lvert z\rvert = 1$	GEV	EV with $\lvert z\rvert > 1$
EL	$-0.5 \leqslant b \leqslant -0.3$ $0.1 \leqslant b \leqslant 0.5$		$-0.2 \leqslant b \leqslant 0.05$	
BX	$-0.5 \leqslant b \leqslant -0.4$ $0.3 \leqslant b \leqslant 0.5$	$b = 0$		$-0.3 \leqslant b < 0$ $0 < b \leqslant 0.2$
ST ($j = 1$)	$-0.5 \leqslant b \leqslant 0.5$			
ST ($j = 2$)	$0 < b \leqslant 0.5$ $b = -0.5$		$b = 0$	$-0.4 \leqslant b < 0$
TA	$-0.5 \leqslant b < 0$ $0 < b \leqslant 0.5$		$b = 0$	

The time-averaged (TA) scheme for $\tilde{v}$ is

$$\tilde{v}_0(t+k)=\tilde{v}_0(t-k)+2\lambda\{\tilde{u}_1(t)-\tfrac{1}{2}[\tilde{u}_0(t-k)+\tilde{u}_0(t+k)]\}$$
$$+2\lambda b\{\tilde{v}_1(t)-\tfrac{1}{2}[\tilde{v}_0(t-k)+\tilde{v}_0(t+k)]\}. \qquad (2.11)$$

The results for LF and TA appear in Table II. (Note that the equations become degenerate with $\kappa_- = -i$, $\kappa_+ = i$, and $|z|=1$ when $b=0$.)

2.3. *Crank–Nicolson*

LF (2.9) can be replaced by the Crank–Nicolson (CN) method

$$[I-\tfrac{1}{2}kAD_0]\,\tilde{w}_v(t+k)=[I+\tfrac{1}{2}kAD_0]\,\tilde{w}_v(t). \qquad (2.12)$$

The results of the numerical verification for CN and either EX (2.4), EL (2.6), BX (2.7), or ST (2.10) are summarized in Table III.

2.4. *Discussion*

For $b=0$, normal-mode analysis can be simplified by using staggered-grid techniques [1]. Chu and Sereny [7] have studied a wave equation in its "natural variables." Sundström [8] proves that EX (2.4) with $j=1$ or 2 or the EL (2.6) is stable with LW (2.1). Other analyses appear in [6, 11] and [12]. The results of this section for $b=0$ agree with these analyses.

For general b, LW is much less sensitive to the choice of the auxiliary boundary condition than either LF (2.9) or CN (2.12); it appears to be the method of choice from the standpoint of stability. LF and CN are only stable with boundary conditions that are insufficiently accurate to insure second-order convergence in the interior. (The generalized eigenvalue for $b=0$, space-time extrapolation (2.10) with

TABLE III

Off-Diagonal Stability Results for Crank–Nicolson

Boundary scheme	Stable	EV with $\|z\|=1$	GEV	EV with $\|z\|>1$
EX ($j=1$)	$-0.5\leqslant b\leqslant 0.5$			
EX ($j=2$)	$-0.5\leqslant b<0$ $0<b\leqslant 0.5$		$b=0$	
EX ($j=3, 4$)	$0<b\leqslant 0.5$	$b=0$		$-0.5\leqslant b<0$
EL	$-0.5\leqslant b\leqslant -0.2$ $0.1\leqslant b\leqslant 0.5$			$-0.1\leqslant b\leqslant 0.05$
BX	$-0.5\leqslant b\leqslant -0.4$ $0<b\leqslant 0.5$	$b=0$		$-0.3\leqslant b<0$
ST ($j=1$)	$-0.5\leqslant b\leqslant 0.5$			

$j=2$, and LF can be removed by adding sufficient dissipation to LF; hence, dissipativity of the interior scheme appears to help.) LW with FLW (2.8) has the minimum accuracy needed for second-order convergence; this combination shows that LW can be sensitive to the choice of boundary condition. Hence, dissipativity or implicitness is insufficient to guarantee stability.

With LF, the stability results indicate that TA (2.11) is the most robust boundary scheme; EL is slightly more stable than BX (2.7) since EL is stable for a wider variety of b values than BX. So, in the sense of boundary condition robustness, TA is better than EL while EL is better than BX. In contrast, Sköllermo suggests that BX is slightly more accurate than EL while EL is more accurate than TA for a scalar problem [22]. Hence, the accuracy and stability of the boundary schemes are conflicting properties. The same conflict occurs for CN with various boundary schemes.

In this section, $u(0, t+k)$ was specified and an auxiliary boundary condition determined $\tilde{v}_0(t+k)$. The determinant stability conditions were of the form

$$\det\begin{bmatrix} 1 & 1 \\ -f(\lambda, \mu_-, \kappa_-, z) & f(\lambda, \mu_+, \kappa_+, z) \end{bmatrix} = 0.$$

If $v(0, t+k)$ were specified and an auxiliary boundary condition used to determine $\tilde{u}_0(t+k)$ then the associated determinant condition would be of the form

$$\det\begin{bmatrix} f(\lambda, \mu_-, \kappa_-, z) & f(\lambda, \mu_+, \kappa_+, z) \\ -1 & 1 \end{bmatrix} = 0.$$

Hence, the stability condition is completely symmetric regardless of whether u or v is given at the $x=0$ boundary.

In an analogous fashion, if A were replaced by $-A$ in the original off-diagonally dominant problem then $R^{-1}(-A)R = \operatorname{diag}(\eta_-, \eta_+)$, where $\eta_\pm = -b \pm 1$ and

$$R = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The determinant conditions would be of the form

$$\det\begin{bmatrix} 1 & 1 \\ f(\lambda, \eta_-, \kappa_-, z) & -f(\lambda, \eta_+, \kappa_+, z) \end{bmatrix} = 0.$$

Note that $\eta_-(b) = \mu_-(-b)$ and $\eta_+(b) = \mu_+(-b)$ (where the notation emphasizes the dependence of the eigenvalues on b). Hence, the stability results would go through with b replaced by $-b$. The existence of an eigenvalue with LW, EX with $j=3$, and $b<0$ is in agreement with the result in [9] for the sign-reversed problem with diagonal entries of $-\frac{1}{2}$.

Only the quarter-space problem ($x \in \mathbb{R}_+$) has been discussed above. Recall that stability of the problems for $x \geqslant 0$ and $x \leqslant 1$ implies stability on the strip $0 \leqslant x \leqslant 1$ [1].

3. Shallow-Water Flows

In this section channel and shallow-water flows are considered. The governing equations are symmetrized variants of the standard meteorological equations; the symmetrized equations are equivalent to the usual ones if the solutions are smooth. The well-posedness of these initial-boundary-value problems is discussed in [23].

3.1. *Channel Flow*

The channel-flow problem with the velocity specified at the boundary is

$$w_t = Aw_x = -\frac{1}{2}\begin{bmatrix} 2u & \phi \\ \phi & 2u \end{bmatrix} w_x \qquad \text{for} \quad (x, t) \in \mathbb{R}_+^2$$

$$w(x, 0) = w_I(x) \qquad \text{for} \quad x \in \mathbb{R}_+$$

$$u(0, t) = d(t) \qquad \text{for} \quad t \in \mathbb{R}_+$$

where $w = (u, \phi)^T$. Here u represents the velocity of the fluid, $\phi = 2\sqrt{gh}$ is the scaled geopotential, and h is the height of the fluid. The usual assumption is that $|u| \ll \phi$.

The associated constant-coefficient problem obtained by freezing the coefficient matrix at some particular values of the solution at a boundary point $(x, t) = (0, t_0)$ can be analyzed. (There are some results based on freezing coefficients for problems with smooth solutions or variable coefficients. In practice, the linear stability theory is applied but the stability of linearized variants does not guarantee the stability of nonlinear problems, in general.) The resulting system of equations is $\hat{w}_t = A_0 \hat{w}_x$, where $A_0 = A(0, t_0)$ and the hats above the variables are used to distinguish the dependent variables of the frozen-coefficient problem. The eigenvalues of A_0 are $\pm \phi_0/2 - u_0$. Hence, the normal-mode theory for discretizations is applicable to the frozen-coefficient problem since it is strictly hyperbolic and the $x = 0$ boundary is not characteristic.

For the frozen-coefficient problem, there are the three natural cases of an inflow ($u_0 > 0$), solid-wall ($u_0 = 0$), and outflow ($u_0 < 0$) boundary. Specifying $\hat{u}$ at the $x = 0$ boundary is reasonable since there is precisely one negative eigenvalue of A_0.

The equations can be rewritten as

$$\hat{w}_t = -\begin{bmatrix} 2u_0/\phi_0 & 1 \\ 1 & 2u_0/\phi_0 \end{bmatrix} \hat{w}_{x'},$$

where x' is x scaled by $\phi_0/2$. The results of Section 2 apply with $-2u_0/\phi_0$ in the role of the parameter b (as noted in the discussion at the close of that section). As an example, discretizing the differential equations with Lax–Wendroff and using linear ($j=2$) extrapolation on $\bar{\phi}$ at the boundary should be stable regardless of the sign of u_0. (See also [6].)

3.2. *Shallow-Water Flow*

Consider a symmetrized shallow-water problem with no y-dependence

$$w_t = -\frac{1}{2}\begin{bmatrix} 2u & 0 & \phi \\ 0 & 2u & 0 \\ \phi & 0 & 2u \end{bmatrix} w_x \quad \text{for} \quad (x, y, t) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$$

$$w(x, y, 0) = w_I(x, y) \quad \text{for} \quad (x, y) \in \mathbb{R}_+ \times \mathbb{R}$$

$$\begin{pmatrix} u \\ a_1 v + a_2 \phi \end{pmatrix}(0, y, t) = d(y, t) \quad \text{for} \quad (y, t) \in \mathbb{R} \times \mathbb{R}_+$$

where $w = (u, v, \phi)^T$, u and v are the x- and y-components of the fluid velocity, respectively, and ϕ is the scaled geopotential. The a_i's are constants used to describe alternate boundary specifications. (The introduction of y-dependence can destabilize the discrete problem [24].)

The differential equations can be broken up into a 2×2 system

$$\begin{pmatrix} u \\ \phi \end{pmatrix}_t = -\frac{1}{2}\begin{bmatrix} 2u & \phi \\ \phi & 2u \end{bmatrix}\begin{pmatrix} u \\ \phi \end{pmatrix}_x$$

and a scalar equation $v_t = -uv_x$. The 2×2 system corresponds exactly to the channel-flow problem discussed earlier. Hence, the stability of the frozen-coefficient variant can be decided.

With an inflow boundary ($u_0 > 0$), the results of Section 2 are directly applicable if $a_1 = 1$ and $a_2 = 0$, that is, if u and v are specified at the boundary then the results of Section 2 can be used. As an example, Lax–Wendroff with the Euler boundary scheme should be stable for the frozen-coefficient inflow case. If there is outflow ($u_0 < 0$), the stability analyses of Section 2 are usable assuming $a_1 = 1$ and $a_2 = 0$ again.

If the boundary conditions are those for solid walls ($a_1 = a_2 = d = 0$) then the normal-mode analysis is not directly applicable since the coefficient matrix is singular at the boundary, that is, the boundary is characteristic. However, the equation for v at the boundary can be dropped since v is initially zero and the coefficient of its governing differential equation is zero along the boundary; this leaves the reduced system for u and ϕ.

4. Gas Dynamics

The initial-boundary-value problem representing the flow of an adiabatic and inviscid gas is

$$w_t = Aw_x = -\begin{bmatrix} u & 0 & \alpha \\ -\alpha & u & 0 \\ p\gamma & 0 & u \end{bmatrix} w_x \qquad \text{for} \quad (x, t) \in \mathbb{R}^2_+$$

$$w(x, 0) = w_I(x) \qquad \text{for} \quad x \in \mathbb{R}_+$$

$$\begin{pmatrix} u \\ a_1 a + a_2 p \end{pmatrix}(0, t) = d(t) \qquad \text{for} \quad t \in \mathbb{R}_+$$

where $w = (u, \alpha, p)^T$. Here u, α, p, and γ denote the velocity, specific volume ($\alpha \equiv \rho^{-1}$), pressure, and ratio of specific heats, respectively. The local sound speed of the gas is $c = \sqrt{p\gamma\alpha}$. Assume that $|u| \ll c$. The well-posedness of this problem is discussed in [23].

If the coefficients of the differential equations are frozen at some boundary point with coefficient matrix A_0 then the resulting problem can be symmetrized. Let

$$S_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \beta_0^{-1} & -\beta_0^{-1} \\ 0 & 0 & \beta_0 \end{bmatrix}$$

where $\beta_0 = \sqrt{p_0\gamma/\alpha_0}$. ($S_0$ is derived from the symmetrizer introduced in [23].) Then

$$\hat{A}_0 = S_0^{-1} A_0 S_0 = -\begin{bmatrix} u_0 & 0 & c_0 \\ 0 & u_0 & 0 \\ c_0 & 0 & u_0 \end{bmatrix}.$$

Hence, the symmetrized equations are $\hat{w}_t = \hat{A}_0 \hat{w}_x$ where $\hat{w} = S_0^{-1} w = (u, \beta_0\alpha + \beta_0^{-1}p, \beta_0^{-1}p)^T$.

The symmetrized equations are of the same form as the shallow-water system discussed in Section 3 so the same stability remarks apply. (See also [25].) Of course, the boundary conditions for the gas-dynamics problem now must be prescribed in terms of the $\hat{w}_i$'s. The $\hat{w}_i$'s are essentially the natural variables except for $\hat{w}_2$; only the specific volume (α) must be replaced by a more complicated quantity. In the solid-wall case ($u_0 = 0$), the stability theory is not directly applicable since the boundary is characteristic but $\hat{w}_2$ will remain zero at the boundary if it is initially zero.

5. Computational Experiments

5.1. *An Off-Diagonally Dominant Problem*

Consider the initial-boundary-value problem given by

$$w_t = Aw_x = \begin{bmatrix} b & 1 \\ 1 & b \end{bmatrix} w_x \quad \text{for} \quad (x, t) \in (0, 1) \times (0, 6]$$

$$w(x, 0) = w_I(x) \quad \text{for} \quad x \in [0, 1]$$

$$u(0, t) = d_l(t) \quad \text{for} \quad t \in (0, 6]$$

$$u(1, t) = d_r(t) \quad \text{for} \quad t \in (0, 6]$$

where $w = (u, v)^T$.

The discrete analog obtained by using Lax–Wendroff and $(j-1)$th-order extrapolation is

$$\tilde{w}_\nu(t+k) = [I + kAD_0 + \tfrac{1}{2}k^2A^2D_+D_-]\,\tilde{w}_\nu(t) \quad \text{for} \quad 0 < \nu < N$$

$$\tilde{w}_\nu(0) = w_I(\nu h) \quad \text{for} \quad 0 \leqslant \nu \leqslant N$$

$$\tilde{u}_0(t+k) = d_l(t+k)$$

$$\tilde{u}_N(t+k) = d_r(t+k)$$

$$D_+^j\,\tilde{v}_0(t+k) = D_-^j\,\tilde{v}_N(t+k) = 0$$

where $h = N^{-1}$. Let $w_I(x) = (\cos(\pi x), \sin(2\pi x))^T$. Finally, take $\lambda = k/h = \frac{1}{2}$, which insures Cauchy stability.

For completeness, define the l_2- and l_∞-norms by $\|\tilde{w}\|^2 = \max_{t \in \Xi} h \sum_{\nu=0}^{N} \|\tilde{w}_\nu(t)\|_2^2$ and $\|\tilde{w}\|_\infty = \max_{t \in \Xi} \max_\nu \|\tilde{w}_\nu(t)\|_\infty$ where $\Xi = \{0.1, 0.2, 0.3, ..., 5.9, 6\}$. Also, let $e_\nu(t) = (w - \tilde{w})_\nu(t)$.

Computer experiments with $b = 0$ and for various values of N and j are summarized in Table IV. Section 2 indicates that cubic ($j = 4$) extrapolation has an eigenvalue with $|z| > 1$, which the computations show dramatically. Both linear ($j = 2$) and quadratic ($j = 3$) extrapolation appear to be stable since second-order convergence is obvious (recall that $\log_{10} 4 \approx 0.6$). Although Section 2 predicts the stability of linear extrapolation, quadratic extrapolation has an eigenvalue with $|z| = 1$. However, such an eigenvalue is unstable because the stability definition [1] includes a sum along the line $x = 0$ for $t \in \mathbb{R}_+$; for a fixed integration interval, $0 \leqslant t \leqslant T$, the instability is not obvious in the computed norms (see [5, 26] and the discussions of Definitions 3.2 and 3.3 in [1]).

The eigenvalue associated with quadratic ($j = 3$) extrapolation moves off of the unit disk as b is made more negative. To illustrate the sensitivity of extrapolation, computations made with $b = -0.05$ are summarized in Table V. The results of computations made with $b = -0.1$ demonstrate even worse behavior. Thus, EX with

TABLE IV

LW and Extrapolation for Off-Diagonal System with $b = 0$[a]

Boundary scheme	N	$\|\tilde{w}\|$	$\|\tilde{w}\|_\infty$	$\log_{10} \|e\|$
EX ($j=2$)	20	1.065	1.818	−1.20
EX ($j=2$)	40	1.022	1.765	−2.00
EX ($j=2$)	80	1.010	1.760	−2.65
EX ($j=3$)	20	1.044	1.778	−1.34
EX ($j=3$)	40	1.020	1.762	−2.02
EX ($j=3$)	80	1.010	1.760	−2.65
EX ($j=4$)	20	3.674 (+06)	10.21 (+06)	6.57
EX ($j=4$)	40	673.7 (+12)	2.907 (+15)	14.8
EX ($j=4$)	80	445.6 (+30)	2.717 (+33)	32.6

[a] The numbers in parentheses represent powers of ten. The computations were performed on IBM 370 equipment in double precision.

$j = 3$ is only marginally stable for $b = 0$ since any negative perturbation causes an instability.

Returning to the case with $b = 0$, extrapolation on $\tilde{v}$ at the left and right boundaries can be replaced by the Euler scheme described in Section 2. Summaries of computations with LW and EL are shown in Table VI. Second-order convergence to the solution can be seen. However, EL does not achieve the same accuracy as EX with $j = 2$ even though EL is expected to have a smaller truncation error [22].

Consider replacing LW with leap-frog. Since LF is a three-level scheme, it is necessary to supply data at $t = k$. One application of LW in the interior with linear ($j = 2$) extrapolation as the auxiliary boundary condition suffices to generate values at $t = k$ from the initial data. This initialization procedure is quite accurate [22].

Table VII summarizes experiments based on LF with either linear ($j = 2$) space-time extrapolation or EL at the boundaries. As the mesh is refined, no convergence to the solution is observed for LF with EL; the error for LF with EL is small until $t = 3$ after which things deteriorate. LF with ST clearly diverges from the correct solution. The generalized eigenvalues associated with these two discretizations are evident from the computations (more on this in the next subsection).

TABLE V

LW and Extrapolation for Off-Diagonal System with $b = -0.05$

B.S.	N	$\|\tilde{w}\|$	$\|\tilde{w}\|_\infty$	$\log_{10} \|e\|$
EX ($j=3$)	20	1.228	1.865	−0.60
EX ($j=3$)	40	1.206	2.536	−0.36
EX ($j=3$)	80	22.67	106.3	1.35

TABLE VI

LW and Euler for Off-Diagonal System with $b=0$

B.S.	N	$\|\tilde{w}\|$	$\|\tilde{w}\|_\infty$	$\log_{10}\|e\|$
EL	20	1.086	1.859	−1.15
EL	40	1.031	1.776	−1.79
EL	80	1.012	1.763	−2.43

5.2. *Channel Flow*

Consider the channel-flow problem with solid-wall boundary conditions given by

$$w_t = -\frac{1}{2}\begin{bmatrix} 2u & \phi \\ \phi & 2u \end{bmatrix} w_x \qquad \text{for} \quad (x, t) \in (0, 1) \times (0, 6]$$

$$w(x, 0) = (0, 0.995 + 10^{-3}[5J_0(5.5x) + \cos(20\pi x)])^T \qquad \text{for} \quad x \in [0, 1]$$

$$u(0, t) = u(1, t) = 0 \qquad \text{for} \quad t \in (0, 6]$$

where $w = (u, \phi)^T$ and J_0 is the zeroth-order Bessel function of the first kind.

This problem can be discretized as before, using local linearization, where $N = h^{-1}$. Let $\lambda = 1$, which insures Cauchy stability for both LW and LF in the frozen-coefficient case. Define $(\|\tilde{w}\|')^2 = \max_{t \in \Xi} h \sum_{\nu=0}^{N-1} \|D_+ \tilde{w}_\nu(t)\|_2^2$. Note that $\|\tilde{w}(0)\|' = 44.68 \times 10^{-3}$ when $N = 80$.

Experiments based on LW are summarized in Table VIII. LW and either EX with $j = 2$ or EL is stable for the channel-flow problem, which concurs with the results of Section 3.

Computations with LF and either ST with $j = 2$ or EL are summarized in Table IX. The experiments show the extreme loss of smoothness due to generalized eigenvalues, even though the l_2- and l_∞-norms of the solution are conserved.

TABLE VII

LF for Off-Diagonal System with $b=0$

B.S.	N	$\|\tilde{w}\|$	$\|\tilde{w}\|_\infty$	$\log_{10}\|e\|$
ST ($j=2$)	20	13.12	27.34	1.12
ST ($j=2$)	40	7.817	16.05	0.89
ST ($j=2$)	80	16.91	40.06	1.23
EL	20	1.115	2.200	−0.47
EL	40	1.036	1.976	−0.79
EL	80	1.021	1.903	−0.79

TABLE VIII

LW for Channel Flow

B.S.	N	$\|\tilde{w}\|$	$\|\tilde{w}\|_\infty$	$\|\tilde{w}\|'$
EX ($j=2$)	20	1.020	1.000	14.53 (–03)
EX ($j=2$)	40	1.008	1.000	28.56 (–03)
EX ($j=2$)	80	1.002	1.001	43.87 (–03)
EL	20	1.020	1.001	16.68 (–03)
EL	40	1.008	1.000	29.22 (–03)
EL	80	1.002	1.001	42.43 (–03)

5.3. *Shallow-Water Flow*

Consider the shallow-water problem with solid-wall boundary conditions given by

$$w_t = Aw_x + Bw_y = -\frac{1}{2}\begin{bmatrix} 2u & 0 & \phi \\ 0 & 2u & 0 \\ \phi & 0 & 2u \end{bmatrix} w_x - \frac{1}{2}\begin{bmatrix} 2v & 0 & 0 \\ 0 & 2v & \phi \\ 0 & \phi & 2v \end{bmatrix} w_y$$

$$\text{for} \quad (x, y, t) \in (0, 1)^2 \times (0, 3]$$

$$w(x, y, 0) = \begin{pmatrix} 0 \\ 0 \\ 0.995 + 10^{-3}[5J_0(5.5x)\sin(2\pi y) + \sin(10\pi x)\sin(10\pi y)] \end{pmatrix}$$

$$\text{for} \quad (x, y) \in [0, 1]^2$$

$$u(0, y, t) = u(1, y, t) = 0 \qquad \text{for} \quad (y, t) \in [0, 1] \times (0, 3]$$

$$v(x, 0, t) = v(x, 1, t) = 0 \qquad \text{for} \quad (x, t) \in [0, 1] \times (0, 3]$$

where $w = (u, v, \phi)^T$.

Redefine $(\|\tilde{w}\|')^2 = \max_{t \in \Xi} h \sum_{\nu_1=0}^{N-1} \sum_{\nu_2=0}^{N-1} \|D_{+x}\tilde{w}_{\nu_1,\nu_2}(t)\|_2^2 + \|D_{+y}\tilde{w}_{\nu_1,\nu_2}\|_2^2$. Note that $\|\tilde{w}(0)\|' = 25.27 \times 10^{-3}$ when $N = 40$.

TABLE IX

LF for Channel Flow

B.S.	N	$\|\tilde{w}\|$	$\|\tilde{w}\|_\infty$	$\|\tilde{w}\|'$
EL	20	1.020	1.002	58.20 (–03)
EL	40	1.008	1.002	103.7 (–03)
EL	80	1.002	1.002	107.4 (–03)
ST ($j=2$)	20	1.020	1.005	131.4 (–03)
ST ($j=2$)	40	1.009	1.017	509.2 (–03)
ST ($j=2$)	80	1.002	1.016	847.8 (–03)

TABLE X

LF and Space-Time Extrapolation for Shallow-Water Flow

B.S.	N	$\|\tilde{w}\|$	$\|\tilde{w}\|_\infty$	$\|\tilde{w}\|'$
ST ($j=2$)	20	1.045	1.003	65.88 (–03)
ST ($j=2$)	40	1.020	1.004	71.72 (–03)

The differential equations can be discretized by the two-dimensional leap-frog scheme

$$\tilde{w}_{\nu_x,\nu_y}(t+k)=\tilde{w}_{\nu_x,\nu_y}(t-k)+2k[AD_{0x}+BD_{0y}]\,\tilde{w}_{\nu_x,\nu_y}(t),$$

where the x's and y's denote differences with respect to the selected variable. Let $h=\Delta x=\Delta y=N^{-1}$. Take $\lambda=\frac{1}{2}$, which insures Cauchy stability for the frozen-coefficient problem. (If A and B were constant matrices, the Cauchy stability limit would be $\lambda(\rho(A)+\rho(B))\leqslant 1$; this restriction can be weakened [27].)

ST with $j=2$ for v and ϕ at the $x=0$ and $x=1$ boundaries suffice as auxiliary boundary conditions. Similarly, ST with $j=2$ on u and ϕ can be used at the $y=0$ and $y=1$ boundaries. Averaging the values obtained from the x- and y-boundary methods suffices to provide values for ϕ at the corners.

Some computations are summarized in Table X. The experiments show that smoothness is being lost rapidly. Freezing the coefficients does not stabilize things so the generalized eigenvalue associated with the one-dimensional problem obtained by ignoring y-dependencies (see Section 3) is being felt. (This differs from [10] where this instability was attributed to two-dimensional effects.) However, LF and ST with $j=2$ are known to be stable for scalar problems in two dimensions [24].

6. Conclusions

Stability analysis of discrete vector initial-boundary-value problems when the boundary conditions are not written in characteristic variables is nontrivial. The examples considered in this paper show that many reasonable interior/boundary difference schemes are not stable. (Characteristic boundary conditions are well understood in terms of scalar problems [13, 14]; the boundary correction procedure described in [14] should be used whenever possible.) Improper boundary treatment can lead to a number of subtle difficulties, as illustrated in Section 5 (see also [28]).

The results of Section 2 suggest that scheme-independent stability criteria [29, 13] will be much more difficult to develop for discretizations written in natural variables. The instability of Lax–Wendroff with the folded Lax–Wendroff scheme demonstrates that dissipativity of the interior method and Cauchy stability of the boundary scheme are not enough.

With the leap-frog or Crank–Nicolson interior scheme, the boundary conditions that are more robust in the sense of stability are less accurate; in other words, the methods that are stable for a wide range of the parameters, λ and b, are less accurate than the schemes that are stable for a narrow range. On the other hand, LW is stable with a variety of boundary conditions so it is the most robust interior scheme from the standpoint of stability.

The stability analyses of the shallow-water and gas-dynamics systems demonstrate the usefulness of studying an off-diagonally dominant problem with two dependent variables.

APPENDIX: NUMERICAL STABILITY VERIFICATION

Normal-mode analysis [1] reduces to solving polynomial systems of the form

$$(\chi_1(\kappa_1, z), \ldots, \chi_m(\kappa_m, z), \Delta(\kappa_1, \kappa_2, \ldots, \kappa_m, z))^T = 0$$

where the χ_j's are the characteristic equations of the interior discretizations and Δ is the determinant condition associated with the boundary schemes.

So we consider the problem of finding all solutions to a polynomial system $\Psi(u) = 0 \in C^n$, where $u \in C^n$. The path-following method described by Garcia and Zangwill [30] is based on a homotopy of the form

$$f(u, t) = (1 - t)\, \Phi(u) + t\Psi(u)$$

where $0 \leqslant t \leqslant 1$ and Φ is an easily solved polynomial system. Let the *realification* of f be denoted by $\bar{f} = (\operatorname{Re} f_1, \operatorname{Im} f_1, \ldots, \operatorname{Re} f_n, \operatorname{Im} f_n)^T$. Then the homotopy defines an ordinary differential equation

$$\bar{f}_{\bar{w}} \frac{d\bar{w}}{ds} = 0 \tag{A.1}$$

where $\bar{w} = (\bar{u}^T, t)^T$, $\bar{f}_{\bar{w}} \equiv \partial\bar{f}/\partial\bar{w}$, and s is the arc-length. It is known that $dt/ds \geqslant 0$ [30] so (A.1) takes solutions of $\Phi(u) = 0$ $(t = 0)$ to a solution of $\Psi(u) = 0$ $(t = 1)$ or to infinity.

The derivative of $\bar{f}$ can be obtained from the derivative of f using the Cauchy–Riemann (CR) equations. In particular, the CR equations suggest the mapping

$$M\colon a + bi \to \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

that gives $\bar{f}_{\bar{w}} = [Mf_u \ (Mf_t)(1, 0)^T] \in \mathbb{R}^{2n \times 2n+1}$.

The system of linear equations for the derivative in (A.1) is underdetermined so a QR factorization with column pivoting is one way to determine a null vector. In the current implementation, the QR routine from LINPACK [31] is used to compute

$Q^T \tilde{f}_{\bar{w}} = [R\ S]$ where Q is orthogonal, P is a permutation matrix, $R \in \mathbb{R}^{2n \times 2n}$ is upper triangular, and $S \in \mathbb{R}^{2n}$. In the full-rank case, a null vector of $\tilde{f}_{\bar{w}} P$ is $(R^{-1}S, -1)^T$. The appropriate null vector is obtained by unscrambling the permutation and scaling the vector so $\|d\bar{w}/ds\|_2 = 1$ and $dt/ds \gtrsim 0$. If $dt/ds \approx 0$ then care must be taken to maintain the correct orientation with respect to the curve. (Gaussian elimination on a $2n \times 2n$ block is often sufficient [32].)

The differential equation (A.1) needs to be integrated from $t = 0$ ($s = 0$) to $t = 1$ or to some large value of s implying that the path is diverging to infinity. The natural procedure is to use an initial-value-problem code that can stop at an implicitly defined point, such as when $t = 1$. The DEROOT code described in [33] has been employed in the present implementation.

The choice of Φ is crucial. Let d_j be the total degree of Ψ_j. Then

$$\Phi(u) = (u_1^{d_1+1} - c_1^{d_1+1}, \ldots, u_n^{d_n+1} - c_n^{d_n+1})^T \tag{A.2}$$

yields finite-length paths to all of the roots of Ψ with the remaining paths diverging to infinity (unless the c_j's are selected from a set of measure zero) [30]. Morgan [32] suggested using $c_j = (j + 0.00143289) + 0.983727i$. Other forms for Φ have been studied that require following $\prod d_j$ paths instead of the $\prod(d_j + 1)$ required by (A.2) [34–36]. One possibility is

$$\Phi(u) = A(u_1^{d_1}, \ldots, u_n^{d_n})^T + c \tag{A.3}$$

where the nonsingular $A \in C^{n \times n}$ and $c \in C^n$ are selected at random [35]. For the computations discussed in Section 2, (A.2) with Morgan's suggested c_j's or (A.3) was used.

When the differential equation solver succeeds in reaching $t = 1$ the solution is improved by a Newton-like method. Rank deficiency must be dealt with.

All of the algorithms are so-called probability one algorithms since the homotopies can break down on sets of measure zero. In particular, ad hoc procedures are required in the leap-frog, time-averaged case when $b \approx 0$ since the usual homotopies break down in that degenerate situation, as mentioned in Section 2. (It is possible to modify the polynomial system to nearly guarantee finite-length paths [37, 36].)

For completeness, I note some of the specifics of the computations that produced Section 2. The various parameters supplied to the DEROOT routine were: (1) a relative error per component between 5×10^{-4} and 10^{-6}; (2) an absolute error of 10^{-8} per component; (3) a relative-error tolerance to locate the root at $t = 1$ between 5×10^{-3} and 10^{-5}. A path was declared to be diverging to infinity if $s > 250$ (or up to 500 if the values were not diverging to infinity) and $t < 1$. The root-finding procedure used at $t = 1$ terminated when a relative error of between 10^{-5} and 10^{-7} was achieved. The stability computations were performed in single precision on a Cray-1A.

Given a root, a determination must be made of whether it is an eigenvalue or a generalized eigenvalue or a root that does not represent an instability. If $|z| \geqslant 1$ and

$\|\kappa\|_\infty < 1$ then the solution is an unstable eigenvalue. On the other hand, if $|z| = 1$ and $\|\kappa\|_\infty = 1$ then a closer examination must be made; if the z is perturbed to be slightly outside of the unit disk and the corresponding $\kappa(z)$'s, determined from the χ's, lie within the disk then a generalized eigenvalue has been found. (Note that the equality and inequality tests mentioned above must be made *fuzzy* to account for the termination criteria of the Newton's method.)

Polynomial resultants are an alternative to homotopy methods. Unfortunately, this approach appears to make the problem harder since univariate polynomials of high degree are the outcome. Another possibility is Drexler's method [38] but the implementation details cannot be deferred to existing mathematical software as with the Garcia–Zangwill algorithm. Finally, there is the possibility of applying Newton-like method at selected values of z near the unit disk as is discussed by Thuné [20]; the provision of an algebraic manipulation facility to generate the polynomial system automatically is commendable. A Newton-like method applied to some of the nonlinear systems described in Section 2 located instabilities but the question arises as to whether a particular starting value can be guaranteed to be in the Newton-attraction region for an eigenvalue or generalized eigenvalue.

The primary disadvantage of homotopy techniques is the relatively high cost. However, homotopy methods are reasonable for studying particular interior/boundary discretizations of an initial-boundary-value problem involving few variables, such as scalar problems.

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